Transport properties of directed percolation clusters at the upper critical dimension

2005 J. Phys.: Condens. Matter 17 S1995

(http://iopscience.iop.org/0953-8984/17/20/022)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **17** (2005) S1995–S2012 doi:10.1088/0953-8984/17/20/022

Transport properties of directed percolation clusters at the upper critical dimension

Olaf Stenull[1] **and Hans-Karl Janssen**[2]

[1] Fachbereich Physik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany
[2] Institut für Theoretische Physik III, Heinrich-Heine-Universität, 40225 Düsseldorf, Germany

Received 30 November 2004, in final form 14 December 2004
Published 6 May 2005
Online at stacks.iop.org/JPhysCM/17/S1995

Abstract

We study the transport properties of directed percolation clusters at the upper critical dimension $d_c = 4 + 1$, where critical fluctuations induce logarithmic corrections to the leading (mean-field) scaling behaviour. Employing field theory and renormalization group methods we calculate these logarithmic corrections up to and including the next to leading correction for a variety of observables, viz. the connectivity, i.e., the probability that two given points are connected, the average two-point resistance and some of the fractal masses describing percolation clusters. Furthermore, we study logarithmic corrections for the multifractal moments of the current distribution on directed percolation clusters.

1. Introduction

Directed percolation (DP) [1, 2] is perhaps the simplest model for directed connectivity in disordered systems. It differs from conventional isotropic percolation (IP) [3] in that activity (in the following electric current) can percolate only along a certain distinguished direction. A simple and intuitive realization of DP is the random diode network (RDN), where nearest neighbouring bonds on a tilted hypercubic lattice are randomly occupied (with probability p) with diodes, see figure 1. Evidently, the effective conductivity along the distinguished direction depends on the occupation probability p. If p is small, all clusters of connected sites are finite so that the conductivity vanishes in the limit of large distances. On the other hand, if p exceeds a certain critical threshold p_c, there is a finite probability to find a directed path between two distant points, leading to a finite resistance when averaged over many independent samples. At the critical threshold $p = p_c$, the system undergoes a continuous phase transition where clusters of conducting paths display a fractal structure. In contrast to IP, these clusters are anisotropic and at criticality they are rather self-affine than self-similar.

In this paper we are interested in the electrical transport properties of DP clusters as realized by the RDN. Due to the self-affinity of the critical clusters, transport quantities like the average resistance $M_R(\mathbf{x}, \mathbf{x}')$ between two connected points $\mathbf{x}$ and $\mathbf{x}'$ exhibit a scaling behaviour as is

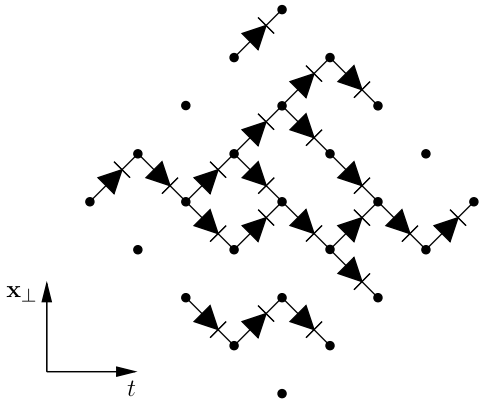

Figure 1. (1 + 1)-dimensional directed bond percolation realized as a random diode network on a tilted square lattice. Electrical current can percolate only along the distinguished direction t.

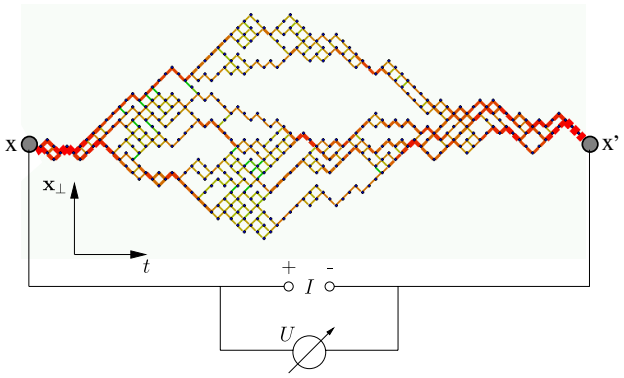

Figure 2. Two-port setup for measuring the resistance of a DP cluster. The figure shows a generic current-carrying backbone when a current I is inserted at a point $\mathbf{x}$ and withdrawn at a point $\mathbf{x}'$. The thickness of the bonds indicates the intensity of the current. The thickest bonds are the so-called red bonds that carry the full current.

(This figure is in colour only in the electronic version)

typical for (anisotropic) critical phenomena. Below the upper critical dimension $d_c = 4 + 1$ this scaling behaviour is characterized by critical exponents which are, due to the effect of fluctuations, anomalous. Above d_c fluctuations are unimportant and the scaling behaviour is purely of mean-field type. Right at d_c, fluctuations lead to logarithmic corrections to the mean-field behaviour. To date, the critical exponents for a variety of transport quantities characterizing DP clusters are known to second order in an ε expansion about d_c, namely the exponents for $M_R(\mathbf{x}, \mathbf{x}')$ [4, 5], for the fractal masses $M_B(\mathbf{x}, \mathbf{x}')$, $M_{red}(\mathbf{x}, \mathbf{x}')$ and $M_{min}(\mathbf{x}, \mathbf{x}')$, respectively, of the backbone, the red (singly connected) bonds and the chemical (shortest) path [4, 6] as well as the exponents for the multifractal moments $M_I^{(l)}(\mathbf{x}, \mathbf{x}')$ of the current distribution [7, 8] (cf figure 2). Logarithmic corrections to these quantities have not previously been studied. With the computer resources and elaborate algorithms available today, computer simulations can explore high dimensions and it seems to be within reach to investigate the transport properties of DP clusters at d_c numerically with good precision. Thus, we feel that it becomes important to investigate the corresponding logarithmic corrections analytically.

The system in statistical physics for which logarithmic corrections have been studied perhaps most thoroughly are linear polymers. The work of Schäfer and co-workers [9, 10] showed that knowing the leading logarithmic corrections is, at least in the case of linear polymers, not sufficient to obtain a satisfactory agreement between theory and simulations. Rather, one has to push the theory beyond the leading corrections, which involves, in addition to the basic analysis of the RG mapping, the calculation of scaling functions. We expect that higher-order logarithmic corrections will be likewise important for the transport properties of DP clusters. In part, this expectation is corroborated by the experience one has with logarithmic corrections in other percolation problems. A comparison between numerical [11] and analytical [12] results for dynamic isotropic percolation at the respective upper critical dimension clearly indicates the importance of the higher order logarithmic corrections. Simulations [13] on dynamical aspects of DP at d_c, on the other hand, show a satisfactory agreement with our theoretic predictions [14] already at the level of the leading logarithmic correction. There are no simulations available yet for a comparison to our analytic results on logarithmic corrections for transport on IP clusters [15].

In this paper we investigate the logarithmic corrections for transport on DP clusters up to and including the next to leading logarithmic correction. For simplicity, we assume that $\mathbf{x} = (\mathbf{0}, 0)$ and $\mathbf{x}' = (\mathbf{0}, t)$. We calculate the quantities of interest as functions of the longitudinal or time-like variable t, that is $M_{\mathrm{R}}(t)$, $M_{\mathrm{B}}(t)$, $M_{\mathrm{red}}(t)$ and $M_{\mathrm{min}}(t)$ as well as $M_I^{(l)}(t)$. By and large, as a byproduct we also obtain the connectivity, i.e., the probability that the points $\mathbf{x} = (\mathbf{0}, 0)$ and $\mathbf{x}' = (\mathbf{0}, t)$ are connected, $P(t)$.

The outline of this paper is as follows. In section 2 we briefly review a generalization of the RDN, viz. the random resistor diode network (RRDN), that has certain advantages over the RDN with respect to setting up a field theoretic model. Then, we explain our field theoretic model, including its variants, and highlight its physical contents. Section 3 reviews renormalization group results for the RRDN in order to establish notation and to provide known results that we need as an input as we go along. Section 4 represents the main part of this paper. Here we derive the desired logarithmic corrections and we state our final results. Concluding remarks are given in section 5. Technical details on the calculation of scaling functions are relegated to an appendix.

2. The model, its variants and the physical contents

2.1. The random resistor diode network

The RRDN introduced by Redner [16–18] is a simple model for electric transport in irregular media that comprises both DP and IP. A RRDN consists of a d-dimensional hyper-cubic lattice in which nearest-neighbour sites are connected by a resistor, a positive diode (conducting only in a preferred direction), a negative diode (conducting only opposite to the preferred direction), or an insulator with respective probabilities p, p_+, p_- and $q = 1 - p - p_+ - p_-$. To be specific, let us choose $\mathbf{n} = 1/\sqrt{d}(1, \ldots, 1)$ as the preferred direction and assume that the bonds $\underline{b}_{\langle i,j\rangle}$ between two nearest neighbouring lattice sites i and j are directed so that $\underline{b}_{\langle i,j\rangle} \cdot \mathbf{n} > 0$. Moreover, let us suppose that the directed bonds obey the generalized Ohm's law

$$\sigma_{\underline{\mathrm{b}}_{\langle i,j\rangle}}(V_{\underline{\mathrm{b}}_{\langle i,j\rangle}})V_{\underline{\mathrm{b}}_{\langle i,j\rangle}} = I_{\underline{\mathrm{b}}_{\langle i,j\rangle}}, \tag{1}$$

where $V_{\underline{\mathrm{b}}_{\langle i,j\rangle}} = V_j - V_i$ is the voltage drop over the bond between sites j and i and $I_{\underline{\mathrm{b}}_{\langle i,j\rangle}}$ is the associated current. The bond conductances $\sigma_{\underline{\mathrm{b}}} = \sigma\,\gamma_{\underline{\mathrm{b}}}$ are random variables where $\gamma_{\underline{\mathrm{b}}}$ takes on the values 1, $\theta(V)$, $\theta(-V)$ (as usual, θ denotes the Heaviside function), and 0 with respective probabilities p, p_+, p_- and q, and where σ is a positive constant. Note that this assumption

means that the diodes are idealized, i.e., under forward-bias voltage they behave as 'ohmic' resistors, whereas they are insulating under backward-bias voltage. Further below we will also deal with nonlinear voltage–current characteristics under forward-bias of the type $V \sim I^r$. The three dimensional phase diagram, i.e., the tetrahedron spanned by the four probabilities, features a non-percolating and three percolating phases, viz. isotropic, positively directed, and negatively directed, with continuous transitions between the four phases. For a detailed discussion of the phase diagram see [16, 17].

The long-length-scale behaviour of the critical RDN and that of the RRDN at either of the two transitions from the non-percolating to the directed percolating phases are equivalent, provided of course, that the distinguished direction of the RDN corresponds to the distinguished direction of the RRDN or its opposite direction, respectively. Thus, we may investigate the transport properties of DP clusters as realized by the RDN via studying the RRDN at either of the two transitions from the non-percolating to the directed percolating phases. We prefer to work with the RRDN because it has certain advantages with respect to setting up a field theoretic model. For details on deriving a field theoretic model for the RRDN we refer to [4, 5]. In the vicinity of the transitions from the non-percolating to either of the directed percolating phases this model can be written in the form of a dynamic response functional [19–21],

$$\mathcal{J} = \int \mathrm{d}^{d_\perp} r \, \mathrm{d}t \left\{ \frac{1}{2} \sum_{\vec{\lambda} \neq \vec{0}} s_{-\vec{\lambda}} \left[\rho \left(\tau - \nabla^2 + w \vec{\lambda}^2 \right) + [\theta(\lambda_0) - \theta(-\lambda_0)] \frac{\partial}{\partial t} \right] s_{\vec{\lambda}} \right.$$
$$\left. + \frac{\rho g}{6} \sum_{\vec{\lambda}, \vec{\lambda}', \vec{\lambda}+\vec{\lambda}' \neq \vec{0}} s_{-\vec{\lambda}} s_{-\vec{\lambda}'} s_{\vec{\lambda}+\vec{\lambda}'} \right\}, \tag{2}$$

whose ingredients have meaning as follows. $s_{\vec{\lambda}} = s_{\vec{\lambda}}(\mathbf{x})$ is an order parameter field which lives on the d-dimensional real space with coordinates $\mathbf{x} = (\mathbf{r}, t)$, where $t \sim x_\parallel = \mathbf{x} \cdot \mathbf{n}$ is the longitudinal part of $\mathbf{x}$ along the preferred direction and $\mathbf{r} = \mathbf{x}_\perp$ is the corresponding $(d_\perp = d - 1)$-dimensional transversal part. In addition to the space coordinate, the order parameter field depends on a D-fold replicated current variable $\vec{\lambda}$ living on a D-dimensional replica space which we will explain a little further below. Regarding the current variable the order parameter field satisfies the constraint $s_{\vec{0}}(\mathbf{x}) = 0$. The parameter τ specifies the distance in phase space to the transition from the non-percolating to the directed phase of interest which occurs in mean field theory at $\tau = 0$. w is proportional to σ^{-1} and ρ is a kinetic coefficient. The meaning of λ_0 will become clear shortly when we elaborate on the replica space.

For regularization purposes, the replica space resembles a discretized D-dimensional torus, i.e., $\vec{\lambda} = \vec{k} \Delta\lambda$ where $\vec{k}$ is a D-dimensional vector with integer components $k^{(\alpha)}$ satisfying $-M < k^{(\alpha)} \leqslant M$ and $k^{(\alpha)} = k^{(\alpha)} \bmod(2M)$. To extract physical quantities from the replica formulation one has to study the limit $D \to 0$, $M \to \infty$ with $(2M)^D \to 1$ and $\Delta\lambda = \lambda_M / \sqrt{M} \to 0$.[3] In addition to these settings dictated by regularization issues, we work near the limit when all the components of $\vec{\lambda}$ are equal and continue to large imaginary values, i.e., we set [22]

$$\lambda^{(\alpha)} = \mathrm{i}\lambda_0 + \xi^{(\alpha)} \tag{3}$$

with real λ_0 and $\xi^{(\alpha)}$ satisfying $|\lambda_0| \gg 1$ and $\sum_{\alpha=1}^{D} \xi^{(\alpha)} = 0$. This specific choice for the replica current is tailored so that: first, we can assign a sign (positive or negative direction) to the multidimensional replica currents. Second, it allows us in actual calculations involving summations or integrations over the replica currents to resort to the saddle point approximation which is crucial since the elementary circuit elements as modelled by equation (1) are nonlinear.

[3] The constant λ_M which sets the width of the voltage interval such that $[-\lambda_M \sqrt{M} < \lambda^{(\alpha)} \leqslant \lambda_M \sqrt{M}]$ is a redundant scaling parameter in the limit $D \to 0$, $M \to \infty$ and hence our predictions are independent of its value.

Finally, we invoke the conditions $\lambda_0^2 \ll D^{-1}$ and $D\lambda_0^2 \ll \vec{\xi}^2 \ll 1$ making $\vec{\lambda}^2 = \vec{\xi}^2 - D\lambda_0^2$ a small positive quantity so that expansions in powers of $\vec{\lambda}^2$ are reliable.

After this excursion into formal aspects which we feel, however, was necessary for a proper definition of the model, we now turn to its physical contents. One of the basic quantities describing the electric transport properties of DP clusters is the average macroscopic resistance

$$M_{\mathrm{R}}(\mathbf{x}, \mathbf{x}') = \frac{\langle \chi_+(\mathbf{x}, \mathbf{x}') R_+(\mathbf{x}, \mathbf{x}') \rangle_C}{P(\mathbf{x}, \mathbf{x}')} \tag{4}$$

when an external current I is inserted at a point $\mathbf{x}$ and extracted at another point $\mathbf{x}'$ provided that the two points are positively connected. In equation (4), $R_+(\mathbf{x}, \mathbf{x}')$ is the total resistance if I is inserted at $\mathbf{x}$ and extracted at point $\mathbf{x}'$ and $\chi_+(\mathbf{x}, \mathbf{x}')$ is an indicator function that gives the value 1 one if $\mathbf{x}$ and $\mathbf{x}'$ are positively connected (if I can percolated from $\mathbf{x}$ to $\mathbf{x}'$), and zero otherwise. $\langle \cdots \rangle_C$ denotes the disorder average over all configurations of the diluted lattice and $P(\mathbf{x}, \mathbf{x}') = \langle \chi_+(\mathbf{x}, \mathbf{x}') \rangle_C$ is the connectivity that measures the probability for $\mathbf{x}$ and $\mathbf{x}'$ being positively connected. The model is set up so that the two-point correlation function

$$G_2(\mathbf{x}, \mathbf{x}', \vec{\lambda}) = \left\langle s(\mathbf{x}, \vec{\lambda}) s(\mathbf{x}', -\vec{\lambda}) \right\rangle \tag{5}$$

is a generating function for $M_{\mathrm{R}}(x, x')$. This can be seen by noting that

$$\begin{aligned} G_2(\mathbf{x}, \mathbf{x}', \vec{\lambda}) &= \left\langle \exp\left[-\frac{\vec{\lambda}^2}{2} R_+(\mathbf{x}, \mathbf{x}') \right] \right\rangle_C \\ &= P(\mathbf{x}, \mathbf{x}') \left\{ 1 - \frac{\vec{\lambda}^2}{2} M_{\mathrm{R}}(\mathbf{x}, \mathbf{x}') + \cdots \right\}, \end{aligned} \tag{6}$$

up to an unimportant multiplicative factor that goes to 1 in the replica limit $D \to 0$. Thus, once we have calculated the two-point correlation function, as we are able by using renormalized field theory, we can extract the average resistance by using

$$M_{\mathrm{R}}(\mathbf{x}, \mathbf{x}') = \frac{\partial}{\partial(-\vec{\lambda}^2/2)} \ln G_2\left(\mathbf{x}, \mathbf{x}', \vec{\lambda}\right)\Big|_{\vec{\lambda}=0}. \tag{7}$$

2.2. The nonlinear RRDN

By generalizing the model we can study much more than just the average resistance. Suppose that the directed bonds obey the nonlinear Ohm's law [23]

$$\sigma_{\underline{\mathrm{b}}_{\langle i,j \rangle}}\left(V_{\underline{\mathrm{b}}_{\langle i,j \rangle}}\right) V_{\underline{\mathrm{b}}_{\langle i,j \rangle}} \left| V_{\underline{\mathrm{b}}_{\langle i,j \rangle}} \right|^{s-1} = I_{\underline{\mathrm{b}}_{\langle i,j \rangle}}. \tag{8}$$

The dynamic functional of the nonlinear RRDN constituted by these elements is of the same form as the functional (2), however, with the replacement [4, 6]

$$w\vec{\lambda}^2 \to w_r \Lambda_r(\vec{\lambda}), \tag{9}$$

where $r = 1/s$ and

$$\Lambda_r(\vec{\lambda}) = -\sum_{\alpha=1}^{D} \left(-\mathrm{i}\lambda^{(\alpha)}\right)^{r+1}. \tag{10}$$

The two-point correlation function in the generalized model has the property that, up to an unimportant constant,

$$\begin{aligned} G_2(\mathbf{x}, \mathbf{x}', \vec{\lambda}) &= \left\langle \exp\left[-\frac{\Lambda_r(\vec{\lambda})}{r+1} R_{r,+}(\mathbf{x}, \mathbf{x}') \right] \right\rangle_C \\ &= P(\mathbf{x}, \mathbf{x}') \left\{ 1 - \frac{\Lambda_r(\vec{\lambda})}{r+1} M_{R_r}(\mathbf{x}, \mathbf{x}') + \cdots \right\}, \end{aligned} \tag{11}$$

where $R_{r,+}(\mathbf{x}, \mathbf{x}')$ is the nonlinear resistance between the two terminals. Thus, the two-point correlation function is here a generating function for the average nonlinear resistance

$$M_{R_r}(\mathbf{x}, \mathbf{x}') = \frac{\langle \chi_+(\mathbf{x}, \mathbf{x}') R_{r,+}(\mathbf{x}, \mathbf{x}') \rangle_C}{P(\mathbf{x}, \mathbf{x}')},$$

which can be calculated by using

$$M_{R_r}(\mathbf{x}, \mathbf{x}') = \frac{\partial}{\partial(-\Lambda_r(\vec{\lambda})/(r+1))} \ln G_2\left(\mathbf{x}, \mathbf{x}', \vec{\lambda}\right)\Big|_{\vec{\lambda}=0}. \tag{12}$$

The generalized RRDN has the benefit that it features the free parameter r and that this parameter can be used to access several physical quantities. For $r \to 1$ one retrieves, of course, the linear average resistance $M_{\mathrm{R}}(\mathbf{x}, \mathbf{x}')$. For $r \to -1^+$, one obtains the mass (the average number of bonds) M_{B} of the backbone,

$$M_{\mathrm{B}}(\mathbf{x}, \mathbf{x}') \sim \lim_{r \to -1^+} M_{R_r}(\mathbf{x}, \mathbf{x}'), \tag{13}$$

as one can see straightforwardly by considering the overall dissipated electric power. Following the lines of Blumenfeld and Aharony [24], one can show that

$$M_{\mathrm{red}}(\mathbf{x}, \mathbf{x}') \sim \lim_{r \to \infty} M_{R_r}(\mathbf{x}, \mathbf{x}'), \tag{14}$$

for the mass of the red bonds and

$$M_{\mathrm{min}}(\mathbf{x}, \mathbf{x}') \sim \lim_{r \to 0^+} M_{R_r}(\mathbf{x}, \mathbf{x}'), \tag{15}$$

for the mass of the chemical path.

2.3. The noisy RRDN

To study multifractal aspects of transport on DP clusters we can generalize the RRDN so that it features static noise. Suppose that the directed bonds have a current–voltage characteristic as stated in equation (1) but that the value of the conductance of occupied bonds under forward bias fluctuates statically about some average value. To model this effect we set $\sigma_{\underline{\mathrm{b}}} = \varsigma_{\underline{\mathrm{b}}} \gamma_{\underline{\mathrm{b}}}$ where $\varsigma_{\underline{\mathrm{b}}}$ is a random variable distributed according to some distribution function f with mean $\overline{\varsigma} = \sigma$ and higher cumulants $\Delta^{(l \geqslant 2)}$ satisfying $\Delta^{(l)} \ll \sigma^l$. The condition on the cumulants is imposed to suppress unphysical negative conductances. The distribution function f might for example be Gaussian, however, our considerations are not limited to this particular choice.

In order to access the multifractal properties of this noisy RRDN we must treat the average $\langle \cdots \rangle_C$ over the configurations of the diluted lattice and the noise average $\{\cdots\}_f$ independently. For this purpose we go beyond the usual replica trick and use a second replication parameter so that the replicated currents $\overleftrightarrow{\lambda}$ are $(D \times E)$ tuples and not merely D tuples [25]. Though the replica space for this model is somewhat more complicated than the one discussed in section 2.1, essentially the same regularization issues appear here and we can make much the same settings and impose much the same conditions as in section 2.1 albeit in a somewhat more general form. For details we refer to [7, 8] where we derived and analysed a field theoretic model for the noisy RRDN. The dynamic functional embodying this model near the transitions from the non-percolating to the directed percolating phases has the same form as the functional (2). However, we have to make the replacement [7, 8]

$$\vec{\lambda} \to \overleftrightarrow{\lambda} \tag{16}$$

and there is an additional term in the dynamic functional,

$$\mathcal{J} \to \mathcal{J} + \sum_{l=2}^{\infty} v_l \mathcal{V}^{(l)}, \tag{17}$$

with dangerously irrelevant interactions

$$\mathcal{V}^{(l)} = \frac{\rho}{2} \int \mathrm{d}^{d_\perp} r \, \mathrm{d}t \, \mathcal{O}^{(l)}(\mathbf{r}, t) \tag{18}$$

with coupling constants $v_l \sim \Delta^{(l)}/\sigma^{2l}$. Here, the $\mathcal{O}^{(l)}$ are composite fields

$$\mathcal{O}^{(l)} = \sum_{\overleftrightarrow{\lambda}} s_{-\overleftrightarrow{\lambda}} K_l(\overleftrightarrow{\lambda}) s_{\overleftrightarrow{\lambda}} \tag{19}$$

featuring the homogeneous polynomials

$$K_l(\overleftrightarrow{\lambda}) = \sum_{\beta=1}^{E} \left[\sum_{\alpha=1}^{D} \left(\lambda^{(\alpha,\beta)} \right)^2 \right]^l. \tag{20}$$

The two-point correlation function $G_2(\mathbf{x}, \mathbf{x}', \overleftrightarrow{\lambda})$ of the noisy RRDN contains a lot of information on the transport properties of DP clusters. In particular it contains the so-called noise cumulants

$$C_{\mathrm{R}}^{(l)}(\mathbf{x}, \mathbf{x}') = \frac{\left\langle \chi_+(\mathbf{x}, \mathbf{x}') \{ R_+(\mathbf{x}, \mathbf{x}')^l \}_f^{(c)} \right\rangle_C}{P(\mathbf{x}, \mathbf{x}')}, \tag{21}$$

where $\{R_+(\mathbf{x}, \mathbf{x}')^l\}_f^{(c)}$ stands for the lth cumulant of the resistance $R_+(\mathbf{x}, \mathbf{x}')$ with respect to f. This can be understood by noting that, up to an unimportant multiplicative constant that goes to one in the replica limit $D \to 0$,

$$\begin{aligned} G_2(\mathbf{x}, \mathbf{x}', \overleftrightarrow{\lambda}) &= \left\langle \exp\left[\sum_{l=1}^{\infty} \frac{(-1/2)^l}{l!} K_l(\overleftrightarrow{\lambda}) \left\{ R_+ \left(\mathbf{x}, \mathbf{x}' \right)^l \right\}_f^{(c)} \right] \right\rangle_C \\ &= P(\mathbf{x}, \mathbf{x}') \left\{ 1 + \sum_{l=1}^{\infty} \frac{(-1/2)^l}{l!} K_l(\overleftrightarrow{\lambda}) \, C_{\mathrm{R}}^{(l)}(\mathbf{x}, \mathbf{x}') + \cdots \right\}, \end{aligned} \tag{22}$$

i.e., the two-point correlation function is a generating function for the noise cumulants which can be extracted by using

$$C_{\mathrm{R}}^{(l)}(\mathbf{x}, \mathbf{x}') = \frac{\partial}{\partial \left[\frac{(-1/2)^l}{l!} K_l(\overleftrightarrow{\lambda}) \right]} \ln G_2(\mathbf{x}, \mathbf{x}', \overleftrightarrow{\lambda}) \Big|_{\overleftrightarrow{\lambda} = \overleftrightarrow{0}}. \tag{23}$$

Although the noise cumulants are interesting in their own right, we are primarily interested in a family of observables that is more intuitive, viz. the multifractal moments

$$M_I^{(l)}(\mathbf{x}, \mathbf{x}') = \frac{\left\langle \chi_+(\mathbf{x}, \mathbf{x}') \sum_{\underline{\mathrm{b}}} \left(I_{\underline{\mathrm{b}}}/I \right)^{2l} \right\rangle_C}{P(\mathbf{x}, \mathbf{x}')} \tag{24}$$

of the current distribution. Using Cohn's theorem [26], one can show that

$$M_I^{(l)}(\mathbf{x}, \mathbf{x}') \sim C_{\mathrm{R}}^{(l)}(\mathbf{x}, \mathbf{x}'). \tag{25}$$

This relation allows us to study the multifractal current distribution indirectly via the noise cumulants, which is important, since, to our knowledge, there is no direct way of calculating the moments of the current distribution by field theoretic means.

3. Renormalization

In recent years we have investigated the scaling behaviour of the average resistance, the fractal masses and the multifractal moments of the current distribution below the upper critical dimension $d_\perp = 4$ [4–8]. In particular we calculated various scaling exponents and fractal dimensions by using renormalized field theory. The aim of this section is to briefly review central elements of our field theory to provide background, to establish notation and to gather previous results that we will need as input as we go on. For background on renormalized field theory in general we refer to [27].

3.1. The linear and the nonlinear RRDN

To keep our presentation compact, we will review here only the general case, i.e., the nonlinear RRDN. The linear RRDN can be retrieved by simply letting $r \to 1$.

As usual, a central building block of our renormalization group analysis is a diagrammatic perturbation theory. Here, the diagrammatic elements for constructing Feynman diagrams are the vertex ρg and the propagator

$$\widetilde{G}(\mathbf{p}, t, \vec{\lambda}) = \widetilde{G}_+(\mathbf{p}, t, \vec{\lambda}) + \widetilde{G}_-(\mathbf{p}, t, \vec{\lambda}), \tag{26}$$

with

$$\widetilde{G}_\pm(\mathbf{p}, t, \vec{\lambda}) = \theta(\pm t)\,\theta(\pm\lambda_0)\exp\left[\mp\rho\left(\tau + \mathbf{p}^2 + w_r\Lambda_r(\vec{\lambda})\right)t\right]\left(1 - \delta_{\vec{\lambda},\vec{0}}\right), \tag{27}$$

where the factor $(1 - \delta_{\vec{\lambda},\vec{0}})$ enforces the constraint $\vec{\lambda} \neq \vec{0}$. For the actual calculations it is sufficient to keep either $\widetilde{G}_+(\mathbf{p}, t, \vec{\lambda})$ or $\widetilde{G}_-(\mathbf{p}, t, \vec{\lambda})$ and we choose to keep $\widetilde{G}_+(\mathbf{p}, t, \vec{\lambda})$.

Calculating the conducting Feynman diagrams in dimensional regularization, one encounters ultraviolet divergences in the form of poles in the deviation $\varepsilon = 4 - d_\perp$ from the upper critical transversal dimension $d_\perp = 4$. These ε poles can be handled by using the renormalization scheme

$$s \to \mathring{s} = Z^{1/2}s, \qquad \rho \to \mathring{\rho} = Z^{-1}Z_\rho\rho, \tag{28a}$$

$$\tau \to \mathring{\tau} = Z_\rho^{-1}Z_\tau\tau, \qquad w_r \to \mathring{w}_r = Z_\rho^{-1}Z_{w_r}w_r, \tag{28b}$$

$$g^2 \to \mathring{g}^2 = Z^{-1}Z_\rho^{-2}Z_u G_\varepsilon^{-1}u\mu^\varepsilon, \tag{28c}$$

where μ^{-1} is the usual arbitrary mesoscopic length scale. The factor $G_\varepsilon = (4\pi)^{-d_\perp/2}\Gamma(1 + \varepsilon/2)$, with Γ denoting the Gamma function, is introduced for convenience. Z, Z_τ, Z_ρ and Z_u are the usual DP Z factors known to second order in ε [28, 29]. In our work on the linear RRDN, see [4, 5], we determined $Z_w = Z_{w_1}$ to second order in ε. Studying the nonlinear RRDN [4, 6], we showed to two-loop order that

$$Z_0 = \lim_{r\to 0} Z_{w_r} = Z, \tag{29a}$$

$$Z_{-1} = \lim_{r\to -1^+} Z_{w_r} = 1, \tag{29b}$$

$$Z_\infty = \lim_{r\to\infty} Z_{w_r} = Z_\tau. \tag{29c}$$

It is not unreasonable to expect that these special relations hold to arbitrary order in perturbation theory, i.e., that they are Ward identities resulting from the symmetries of the system. Identifying these symmetries and proving the presumable Ward identities is a challenging open issue for future work. Here, for our calculations to follow below we need to know the

renormalization factors explicitly only to one-loop order,

$$Z = 1 + \frac{u}{4\varepsilon}, \qquad Z_\rho = 1 + \frac{u}{8\varepsilon}, \tag{30a}$$

$$Z_\tau = 1 + \frac{u}{2\varepsilon}, \qquad Z_u = 1 + \frac{2u}{\varepsilon}, \tag{30b}$$

$$Z_{w_r} = 1 + \frac{u}{2\varepsilon}\left(1 - \frac{1}{2^{r+1}}\right). \tag{30c}$$

The fact that the unrenormalized theory must be independent of the inverse length scale introduced in the renormalization process can be used in a routine fashion to set up a Gell-Mann–Low renormalization group equation (RGE) for the correlation functions. For an N point function this RGE reads

$$\left[\mathcal{D}_{\mu,r} + \frac{N}{2}\gamma\right] G_N\left(\left\{\mathbf{r}, \rho t, w_r \Lambda_r(\vec{\lambda})\right\}; \tau, u, \mu\right) = 0, \tag{31}$$

where

$$\mathcal{D}_{\mu,r} = \mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial u} + \tau\kappa\frac{\partial}{\partial\tau} + w_r\zeta_r\frac{\partial}{\partial w_r} + \rho\zeta_\rho\frac{\partial}{\partial\rho}. \tag{32}$$

The functions featured in the RGE are given to two-loop order by

$$\beta(u) = -\varepsilon u + \frac{3u^2}{2} - \left(169 + 106\ln\frac{4}{3}\right)\frac{u^3}{128} + \mathrm{O}(u^4), \tag{33a}$$

$$\kappa(u) = \frac{3u}{8} - \left(7 + 10\ln\frac{4}{3}\right)\frac{7u^2}{256} + \mathrm{O}(u^3), \tag{33b}$$

$$\gamma(u) = -\frac{u}{4} + \left(6 - 9\ln\frac{4}{3}\right)\frac{u^2}{32} + \mathrm{O}(u^3), \tag{33c}$$

$$\zeta_\rho(u) = -\frac{u}{8} + \left(17 - 2\ln\frac{4}{3}\right)\frac{u^2}{256} + \mathrm{O}(u^3), \tag{33d}$$

$$\zeta_r(u) = \zeta_{r,1}u + \zeta_{r,2}u^2 + \mathrm{O}(u^3). \tag{33e}$$

The coefficient of the first order term in equation (33a) is known for arbitrary r,

$$\zeta_{r,1} = \left(\frac{3}{8} - \frac{1}{2^{r+2}}\right).$$

The coefficient $\zeta_{r,2}$ is known only for particular values of r. Our two-loop analysis of the linear RRDN with $r = 1$ gave

$$\zeta_{1,2} = -\tfrac{5}{32}. \tag{34}$$

For the remaining values of r of interest here, $\zeta_{r,2}$ can be inferred readily from the relations

$$\zeta_{-1}(u) = \gamma(u) - \zeta_\rho(u), \qquad \zeta_0(u) = -\zeta_\rho(u), \tag{35a}$$

$$\zeta_\infty(u) = \kappa(u) \tag{35b}$$

stemming from the special relations between the Z-factors.

In the following we will use for the RGE-functions an abbreviated notation of the type $f(u) = f_1u + f_2u^2 + \cdots$. For example, we will write equation (33a) as $\beta(u) = \beta_1u + \beta_2u^2 + \beta_3u^3 + \mathrm{O}(u^4)$ and likewise for the other RGE-functions.

3.2. The noisy RRDN

The parameters v_l featured in the dynamic functional of the noisy RRDN are, in contrast to the relevant parameter $w = w_1$, dangerously irrelevant, i.e., they are irrelevant on dimensional grounds but they must not be neglected in studying the noise cumulants because we otherwise inevitably lose the information we are interested in. Due to their irrelevance, the v_l cannot be treated in the same fashion as the relevant w. A proper treatment of the v_l can be achieved by looking at insertions of the dangerously irrelevant interactions $\mathcal{V}^{(l)}$ into Feynman diagrams. Due to the irrelevance, insertions of $\mathcal{V}^{(l)}$ generate a multitude of terms corresponding to interactions with equal or lower naive dimension than $\mathcal{V}^{(l)}$. All these interactions have to be taken into account in the renormalization process. The interactions of lower naive dimension, however, merely lead to subdominant corrections and can be ignored for our purposes. Keeping all the interactions of the same naive dimension, we have a renormalization in matrix form

$$\underline{\mathcal{V}}^{(l)} \to \mathring{\underline{\mathcal{V}}}^{(l)} = \left(\underline{\underline{Z}}^{(l)}\right)^{-1} \underline{\mathcal{V}}^{(l)}, \tag{36}$$

where $\underline{\mathcal{V}}^{(l)} = (\mathcal{V}^{(l)}, \mathcal{V}_2^{(l)}, \ldots)$ is a vector that contains all the interactions generated by $\mathcal{V}^{(l)}$ by the renormalization process including $\mathcal{V}^{(l)}$ itself. The $\mathcal{V}^{(l)}$ are distinguished by the feature that the interactions generated by $\mathcal{V}^{(l)}$ do not in turn generate $\mathcal{V}^{(l)}$, or, as we say, that the corresponding composite fields $\mathcal{O}^{(l)}$ are master operators [30, 31, 7, 8]. Because of their subordinate role, we refer to the remaining operators $\mathcal{O}_2^{(l)}$ and so on as servants. The master operators are associated with renormalization matrices $\underline{\underline{Z}}^{(l)} = \underline{\underline{1}} + \mathrm{O}(u)$, where $\underline{\underline{1}}$ stands for the unit matrix, of a particularly simple structure

$$\underline{\underline{Z}}^{(l)} = \begin{pmatrix} Z^{(l)} & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}, \tag{37}$$

where the $*$ symbolize arbitrary elements. As a consequence of the simple structure of $\underline{\underline{Z}}^{(l)}$ the servants can be neglected in calculating the scaling index of their master. This can be seen as follows. Due to equation (23) we are ultimately interested in derivatives of the Green function with insertions of $\mathcal{V}^{(l)}$, $G_{2;\mathcal{V}^{(l)}}$, with respect to $K_l(\overleftrightarrow{\lambda})$ evaluated at $\overleftrightarrow{\lambda} = 0$. From the renormalization of the master interaction,

$$\mathcal{V}^{(l)} = Z^{(l)} \mathring{\mathcal{V}}^{(l)} + \sum_{\alpha \geqslant 2} Y_\alpha^{(l)} \mathring{\mathcal{V}}_\alpha^{(l)}, \tag{38}$$

where the $Y_\alpha^{(l)}$ are elements of $\underline{\underline{Z}}^{(l)}$, it follows that

$$G_2(\mathbf{r}, t)_{\mathcal{V}^{(l)}} = Z^{(l)} G_2(\mathbf{r}, t)_{\mathring{\mathcal{V}}^{(l)}} + \sum_{\alpha \geqslant 2} Y_\alpha^{(l)} G_2(\mathbf{r}, t)_{\mathring{\mathcal{V}}_\alpha^{(l)}}. \tag{39}$$

The coefficients $Y_\alpha^{(l)}$ pertaining to the servants are required to make $G_2(\mathbf{r}, t)_{\mathcal{V}^{(l)}}$ free of ε-poles. However, only the first term on the right-hand side of equation (39) gives a non-zero contribution when we differentiate with respect to $K_l(\overleftrightarrow{\lambda})$ and then set $\overleftrightarrow{\lambda} = 0$ since only the insertion of $\mathring{\mathcal{V}}^{(l)}$ produces the polynomial structure of $K_l(\overleftrightarrow{\lambda})$. Hence, as long as we restrict ourselves to the properties of the noise cumulants $C_{\mathrm{R}}^{(l)}(\mathbf{x}, \mathbf{x}')$ we only need for our practical purposes the multiplicative renormalizations of the $\mathcal{V}^{(l)}$, i.e., we only need the element $Z^{(l)}$. We can set all the other renormalizations $Y_\alpha^{(l)}$ pertaining to the servants formally to zero.

Understanding that the renormalization of the dangerously irrelevant interactions $\mathcal{V}^{(l)}$ is subtle but that we can ignore these subtleties for our practical purposes, we treat their couplings

Table 1. The coefficients $\gamma_2^{(l)}$ appearing in equation (43).

l	2	3	4	5
$\gamma_2^{(l)}$	−0.242 49	−0.265 23	−0.269 88	−0.270 45

v_l in much the same way as w. Doing so we have to bear in mind, of course, that this procedure only makes sense if we expand all Feynman diagrams in powers of v_l and truncate this expansion after linear order (which corresponds to using single insertions). Then, we renormalize the v_l by setting

$$v_l \to \mathring{v}_l = Z_\rho^{-1} Z_{v_l} v_l = Z^{(l)} v_l \tag{40}$$

Z_{v_l} is known for arbitrary $l = 0, 1, 2, \ldots$ to one-loop order and for most important values of l to two-loop order. Below we will need to know Z_{v_l} explicitly to one-loop order for calculating the desired logarithmic corrections. To this order Z_{v_l} is related to the Z_{w_r} by

$$Z_{v_l} = Z_{w_{2l-1}} = 1 + \left(1 - \frac{1}{4^l}\right)\frac{u}{2\varepsilon} \tag{41}$$

as we show in the appendix. The two-point correlation function is now governed by the RGE

$$\left\{\left[\mathcal{D}_{\mu,1} + \gamma\right] + \sum_l \gamma^{(l)} v_l \frac{\partial}{\partial v_l}\right\} G_2\left(\mathbf{r}, \rho t, w\overset{\leftrightarrow}{\lambda}^2; \{v_l K_l(\overset{\leftrightarrow}{\lambda})\}, \tau, u, \mu\right) = 0. \tag{42}$$

The Gell-Mann–Low function $\gamma^{(l)}$ stemming from $Z^{(l)}$ is

$$\gamma^{(l)} = \gamma_1^{(l)} u + \gamma_2^{(l)} u^2 + \mathrm{O}(u^3), \tag{43}$$

where

$$\gamma_1^{(l)} = \zeta_{2l-1,1} = \frac{3}{8} - \frac{1}{2^{2l+1}}. \tag{44}$$

$\gamma^{(l)}$ satisfies the relations $\gamma^{(0)} = \zeta_{-1} = \gamma - \zeta_\rho$ and $\gamma^{(1)} = \zeta_1$. $\gamma_2^{(l)}$ is stated for $l = 2, \ldots, 5$ in table 1.

4. Logarithmic corrections

The RGE can be solved by the method of characteristics whereby one introduces a single flow parameter ℓ and sets up characteristic equations that describe how the scaling parameters transform under a change of ℓ. The characteristic for the momentum scale μ is particularly simple and has the solution $\bar{\mu}(\ell) = \mu\ell$, i.e., a change of ℓ corresponds to a change of the external inverse length scale. With the help of the solution to the remaining characteristics one obtains

$$\begin{aligned} G_2(\mathbf{r}, \rho t, w_r \Lambda_r(\lambda); \{v_l K_l(\lambda)\}, \tau, u, \mu) &= (\mu\ell)^{d_\perp} \bar{Z}(\ell) \\ &\times G_2\left(\mu\ell\,\mathbf{r}, (\mu\ell)^2 \bar{\rho}(\ell)\, t, \frac{\bar{w}_r(\ell)\,\Lambda_r(\lambda)}{(\mu\ell)^2}; \left\{\frac{\bar{v}_l(\ell)\,K_l(\lambda)}{(\mu\ell)^2}\right\}, \frac{\bar{\tau}(\ell)}{(\mu\ell)^2}, \bar{u}(\ell), 1\right) \end{aligned} \tag{45}$$

as a solution to the RGE. This formula applies to the nonlinear as well as to the noisy RRDN with the understanding that we have to set $v_l = 0$ and $\lambda = \vec{\lambda}$ in the former and $r = 1$ and $\lambda = \overset{\leftrightarrow}{\lambda}$ in the latter case. At this stage the scaling solution (45) is still rather formal since $\bar{Z}(\ell)$, $\bar{\rho}(\ell)$, $\bar{\tau}(\ell)$, $\bar{w}_r(\ell)$, $\bar{v}(\ell)$ and $\bar{u}(\ell)$ require specification. Below the upper critical

dimension, these quantities display power law behaviour for $\ell \to 0$ described by the well known critical exponents of the DP universality class, the resistance exponents ϕ_r, and the multifractal exponents ψ_l of the RRDN. Directly in $d_\perp = 4$, they depend logarithmically on ℓ and hence their behaviour is qualitatively different from the lower dimensional case.

Now we will state and solve the characteristics directly for $d_\perp = 4$. The characteristic for the dimensionless coupling constant u is given by

$$\ell \frac{\mathrm{d}\bar{u}}{\mathrm{d}\ell} = \beta(\bar{u}). \tag{46}$$

The solution to this differential equation for $\varepsilon = 0$ is

$$\ell = \ell(\bar{u}) = \ell_0 \bar{u}^{-\beta_3/\beta_2^2} \exp\left[-\frac{1}{\beta_2 \bar{u}} + \mathrm{O}(\bar{u})\right], \tag{47}$$

where ℓ_0 is an integration constant. The remaining characteristics are all of the same structure, namely

$$\ell \frac{\mathrm{d} \ln \bar{Q}(\bar{u})}{\mathrm{d}\ell} = q(\bar{u}), \tag{48}$$

where Q is a placeholder for Z, ρ, τ, w_r and v_l, respectively, and q is a placeholder for γ, ζ_ρ, κ, ζ_r and $\gamma^{(l)}$, respectively. Exploiting $\ell\, \mathrm{d}/\mathrm{d}\ell = \beta\, \mathrm{d}/\mathrm{d}\bar{u}$ one obtains the solution

$$\bar{Q}(\bar{u}) = Q_0 \bar{u}^{q_1/\beta_2} \exp\left[\frac{(q_2\beta_2 - q_1\beta_3)}{\beta_2^2}\bar{u} + \mathrm{O}(\bar{u}^2)\right], \tag{49}$$

where Q_0 is a non-universal integration constant.

After having reviewed some of the cornerstones of the field theory of the RRDN we will now determine the critical behaviour of the connectivity, the average resistance, the fractal masses of the backbone, the chemical path and the red bonds as well as the multifractal moments of the current distribution at the upper critical dimension $d_\perp = 4$. For simplicity, we set in the following $\mathbf{x}' = (\mathbf{r}', 0) = (\mathbf{0}, 0)$ and $\mathbf{x} = (\mathbf{r}, 0) = (\mathbf{0}, t)$ and restrict our attention to the behaviour of the aforementioned quantities as functions of the time-like variable t. To this end we choose

$$(\mu\ell)^2 \bar{\rho}(\ell) t = X_0, \tag{50}$$

where X_0 is a constant of order 1. With this choice $\bar{u}$ and ℓ tend to zero for $\rho\mu^2 t \to \infty$. From equations (47) and (49), specialized to $\bar{\rho}$, we obtain with $\beta_2 = 3/2$

$$t = t_0 \bar{u}^{-2a/3} \exp\left(\frac{4}{3\bar{u}}\right)[1 + \mathrm{O}(\bar{u})], \tag{51}$$

where $t_0 = X_0/(\mu^2 \ell_0^2\, \rho_0)$ is yet another non-universal constant and a is given by

$$a = \frac{\beta_2 \zeta_{\rho,1} - 2\beta_3}{2\beta_2} = \frac{157}{192} + \frac{53}{96} \ln\frac{4}{3} = 0.976\,53. \tag{52}$$

At certain stages it will be more convenient to use, instead of using the original t, the variable

$$s = \tfrac{3}{4} \ln(t/t_0). \tag{53}$$

For s, equation (51) translates into

$$s = \bar{u}^{-1} - a \ln \bar{u} + \mathrm{O}(\bar{u}). \tag{54}$$

Finally, we find by using equation (54)

$$\bar{u} = s^{-1} \exp\left[a\frac{\ln s}{s} + \mathrm{O}\left(\frac{\ln^2 s}{s^2}, \frac{\ln s}{s^2}, \frac{1}{s^2}\right)\right] \tag{55}$$

for the dimensionless coupling constant as a function of the longitudinal coordinate s.

At this point we know the general scaling form (45) of the two-point correlation function and thus we have an important part of the information that we need to understand the critical behaviour of connectivity, the fractal masses and the multifractal moments at the upper critical dimension. However, for determining the logarithmic corrections to the scaling behaviour of these quantities beyond the leading correction, we need to know some additional information about the scaling function appearing on the right-hand side of equation (45).

Expanding the right-hand side of equation (45) at criticality $\tau = 0$ to linear order in $\bar{w}_r \Lambda_r$ and $\bar{v}_l K_l$, and by using equation (50) we find that the two-point correlation function is at $d_\perp = 4$ of the form

$$G_2(\mathbf{0}, \rho t, w_r \Lambda_r(\lambda); \{v_l K_l(\lambda)\}, 0, u, \mu) = (\mu\ell)^4 \bar{Z}(\bar{u}) G_2(\mathbf{0}, X_0, 0; \{0\}, 0, \bar{u}, 1) \times \Big\{ 1 + (\mu\ell)^{-2} \bar{w}_r(\bar{u}) \Lambda_r(\lambda) g_2'(X_0, \bar{u}) + (\mu\ell)^{-2} \sum_l \bar{v}_l(\bar{u}) K_l(\lambda) g_2^{(l)}(X_0, \bar{u}) \cdots \Big\}, \tag{56}$$

with expansion coefficients $g_2'(X_0, \bar{u}) = \partial \ln G_2 / \partial w_r \Lambda_r|_{\lambda=0}$ and $g_2^{(l)}(X_0, \bar{u}) = \partial \ln G_2 / \partial v_l K_l|_{\lambda=0}$. For the second logarithmic correction we have to calculate the functions $G_2(\mathbf{0}, X_0, 0; \{0\}, 0, \bar{u}, 1)$, $g_2'(X_0, \bar{u})$ and $g_2^{(l)}(X_0, \bar{u})$ to one-loop order. Some details on these calculations can be found in the appendix.

4.1. Connectivity, average resistance and fractal masses

Our one-loop calculation sketched in the appendix yields that the scaling functions relevant for the connectivity, the average resistance and the fractal masses are given by

$$G_2(\mathbf{0}, X_0, 0; \bar{u}, 0, 1) = c_0 \big[1 + A_P \bar{u} + \mathrm{O}(\bar{u}^2) \big], \tag{57a}$$

$$g_2'(X_0, \bar{u}) = c_1 \big[1 + A_r \bar{u} + \mathrm{O}(\bar{u}^2) \big], \tag{57b}$$

where c_0 and c_1 are non-universal constants and where A_P and A_r are amplitudes that can depend on X_0. A_P turns out being independent of X_0,

$$A_P = \tfrac{3}{16}. \tag{58}$$

For the amplitude A_r we find

$$A_r = \frac{1 - \ln 2 - \mathcal{Z}}{8} \left(1 - \frac{1}{2^r} \right), \tag{59}$$

where $\mathcal{Z} = \mathcal{Z}(X_0)$ is a non-universal constant that we have defined, to stay consistent with our work on logarithmic corrections for dynamic properties of DP [14], as $\mathcal{Z}(X_0) = C_E + \ln(2X_0)$ with $C_E = 0.577\,215\ldots$ being Euler's constant. Alternatively, it is possible to choose the arbitrary parameter X_0 in such a way that $\mathcal{Z} = 1 - \ln 2$ so that the amplitude A_r becomes simply zero. Physically, this would only change the non-universal timescale t_0 entering the scaling functions through equation (51). However, we prefer to choose $\mathcal{Z}$ as we did for two reasons. First, as mentioned above, this allows us to stay consistent with previous work. Second, leaving $\mathcal{Z}$ in our formulas has the advantage that $\mathcal{Z}$ can be used as a fit parameter when it comes to comparing our results to simulations.

Now we are finally in the position to assemble our results for the connectivity and the average (nonlinear) resistance. Merging equations (11), (56) and (57*a*) we obtain for the connectivity

$$P(t) \sim (\mu\ell)^4 \bar{Z}(\bar{u}) \big[1 + A_P \bar{u} + \mathrm{O}(\bar{u}^2) \big]. \tag{60}$$

Exploiting our choice (50) and using our knowledge about the solutions of the characteristics we find

$$t^2 P(t)/P_0 = \left(1 + \frac{3\bar{u}}{16}\right) \exp(-c_P \bar{u}) \left[1 + \mathrm{O}(\bar{u}^2)\right] \tag{61a}$$

$$= \left(1 + \frac{3}{16s}\right) \left\{1 - \frac{c_P}{s} + \mathrm{O}\left(\frac{\ln^2 s}{s^2}, \frac{\ln s}{s^2}, \frac{1}{s^2}\right)\right\}, \tag{61b}$$

where P_0 is a non-universal constant and

$$c_P = \frac{2\zeta_{\rho,2} - \gamma_2}{\beta_2} = -\frac{1}{192}\left(7 - 34 \ln \frac{4}{3}\right) = 0.014\,48. \tag{62}$$

Equation (61*a*) in conjunction with equation (51) can be viewed as a parametric representation of the tuple $(t^2 P(t), t)$ with $\bar{u}$ serving as the free parameter. Equation (61*b*) states our result in a more traditional form. Though perhaps less intuitive, the parametric form has the conceptual advantage that it involves only one expansion variable, viz. the effective coupling constant $\bar{u}$, whereas in the traditional form functions of the longitudinal variable such as $1/s^2$, $\ln s/s^2$, $\ln^2 s/s^2$, and so on, compete against each other. Note that the usually leading logarithmic correction is absent in case of the connectivity $P(t)$. Hence, one has to go to the next order, as we did, to see a deviation from the mean-field field behaviour $P(t) \propto t^{-2}$.

Next we turn to the average nonlinear resistance. Using equations (6), (45), (56), (57*b*) as well as equation (12) yields

$$M_{R_r}(t) \sim (\mu\ell)^{-2} \bar{w}_r(\bar{u}) \big[1 + A_r \bar{u} + \mathrm{O}(\bar{u}^2)\big]. \tag{63}$$

Inserting several intermediate results we are led to

$$t^{-1} M_{R_r}(t)/M_{R_r,0} = \big(\bar{u}^{-1} + B\big)^{a_r} \exp(-c_r \bar{u}) \left[1 + \mathrm{O}(\bar{u}^2)\right] \tag{64a}$$

$$= \big(s + B\big)^{a_r} \left\{1 - \frac{b_r \ln s + c_r}{s} + \mathrm{O}\left(\frac{\ln^2 s}{s^2}, \frac{\ln s}{s^2}, \frac{1}{s^2}\right)\right\} \tag{64b}$$

with a non-universal constant $M_{R_r,0}$ and

$$a_r = -\frac{\zeta_{\rho,1} + \zeta_{r,1}}{\beta_2} = -\frac{1 - 2^{-r}}{6}, \tag{65a}$$

$$b_r = a a_r, \tag{65b}$$

$$c_r = \frac{(\zeta_{\rho,1} + \zeta_{r,1})\beta_3}{\beta_2^2} - \frac{\zeta_{\rho,2} + \zeta_{r,2}}{\beta_2}, \tag{65c}$$

$$B = \frac{A_r}{a_r} = -\frac{3}{4}\big(1 - \ln 2 - \mathcal{Z}\big). \tag{65d}$$

As explained above, the physically most important limits of r are $r \to 1, -1^+, \infty, 0^+$ since these limits yield, respectively, the average linear resistance and the fractal masses of the backbone, the red bonds and the chemical path. The numerical values of b_r and c_r in the limits $r \to 1, -1^+, \infty$ are collected in table 2. In the limit $r \to 0^+$ the quantities a_r, b_r, c_r, and A_r vanish and the amplitude B does not make much sense. This vanishing of the parameters had to be expected because in DP the length of the chemical path between the two points $(\mathbf{0}, 0)$ and $(\mathbf{0}, t)$ is essentially t and that hence

$$M_{\min} \propto t \tag{66}$$

with no logarithmic correction. The fact that our result is in conformity with this anticipated behaviour is reassuring and we rate it as an important consistency check for our calculation.

Table 2. Values of the numbers b_r and c_r appearing in equations (64).

r	-1	1	∞
b_r	0.162 75	−0.081 37	−0.162 75
c_r	0.102 11	−0.025 19	−0.035 89

Table 3. Values of the numbers $b^{(l)}$ and $c^{(l)}$, appearing in equations (69).

l	2	3	4	5
$b^{(l)}$	−0.142 41	−0.157 66	−0.161 48	−0.162 43
$c^{(l)}$	−0.032 63	−0.033 71	−0.034 66	−0.035 30

4.2. Multifractal moments

The logarithmic corrections for the multifractal moments can be calculated in much the same way as the corrections for the average nonlinear resistance. Our one-loop calculation sketched in the appendix yields

$$g_2^{(l)}(X_0, \bar{u}) = c_1\big[1 + A_{2l-1}\,\bar{u} + \mathrm{O}(\bar{u}^2)\big], \tag{67}$$

where the amplitude A_{2l-1} can be read off from equation (59) setting $r = 2l - 1$. Using equations (22), (25), (56), (67) as well as (23) and equation (67) we obtain

$$M_I^{(l)}(t) \sim (\mu\ell)^{-2}\bar{v}^{(l)}(\bar{u})\big(1 + A_{2l-1}\bar{u} + \mathrm{O}(\bar{u}^2)\big). \tag{68}$$

Recalling our choice for the flow parameter, equation (50), and using that the solution of the characteristic for $\bar{v}^{(l)}$ is of the form (49) we obtain

$$t^{-1}\,M_I^{(l)}(t)/M_{I,0}^{(l)} = \big(\bar{u}^{-1} + B\big)^{a^{(l)}} \exp(-c^{(l)}\,\bar{u})\,\big[1 + \mathrm{O}(\bar{u}^2)\big] \tag{69a}$$

$$= \big(s + B\big)^{a^{(l)}}\left\{1 - \frac{b^{(l)}\ln s + c^{(l)}}{s} + \mathrm{O}\left(\frac{\ln^2 s}{s^2}, \frac{\ln s}{s^2}, \frac{1}{s^2}\right)\right\} \tag{69b}$$

where $M_{I,0}^{(l)}$ are non-universal constants. The amplitude B is the same as the one in equation (65). The parameters $a^{(l)}$ and $b^{(l)}$ are related to the parameters of equations (65) by

$$a^{(l)} = a_{2l-1}, \qquad b^{(l)} = b_{2l-1}, \tag{70}$$

and $c^{(l)}$ is given by

$$c_I^{(l)} = \frac{(\zeta_{\rho,1} + \gamma_1^{(l)})\beta_3}{\beta_2^2} - \frac{\zeta_{\rho,2} + \gamma_2^{(l)}}{\beta_2}\,. \tag{71}$$

Numerical values for the coefficient $c_I^{(l)}$ can be found in table 3. Note that $M_I^{(1)} \sim C_{\mathrm{R}}^{(1)} = M_{\mathrm{R}}$ as it should. Note also that $M_I^{(0)} \sim C_{\mathrm{R}}^{(0)} = M_{\mathrm{B}}$. This must hold since $\sum_{\mathrm{b}}(I_{\mathrm{b}}/I)^{2l}$ coincides for $l \to 0$ with the number of current-carrying bonds and thus $M_I^{(0)} = M_{\mathrm{B}}$.

5. Concluding remarks

In summary, we have studied the electrical transport properties of DP clusters by using the methods of renormalized field theory. To our knowledge, logarithmic corrections for the connectivity, the average resistance, the fractal masses of the backbone, the red bonds, the chemical path and the multifractal moments of the current distribution have not previously been considered. We calculated these corrections up to and including the next to leading correction.

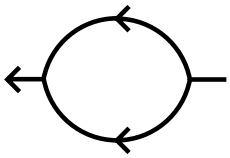

Figure A.1. Diagrammatic representation of the self-energy $\Sigma(\mathbf{q}, t, \lambda)$ at one-loop order.

We hope that our work triggers complementary numerical simulations. With today's computer hardware and sophisticated algorithms, our results should be testable by numerical work. Because we went beyond just calculating the leading corrections, we are optimistic that our results compare well with simulations, perhaps even quantitatively.

Appendix. Amplitudes

In this appendix we deliver some details on our calculation of the amplitudes entering the second logarithmic corrections. First, let us look at the two-point correlation function at zero-loop level. As a function of the longitudinal and the transversal coordinates we have

$$G_2^{(0)}(\mathbf{r}, t, \lambda) = \int_{\mathbf{p}} \exp(\mathrm{i}\mathbf{p}\cdot\mathbf{r})\widetilde{G}_+(\mathbf{p}, t, \lambda), \tag{A.1}$$

where $\int_{\mathbf{p}}$ is an abbreviation for $1/(2\pi)^{d_\perp}\int \mathrm{d}^{d_\perp}p$. To treat the nonlinear and the noisy RRDN in one go, we use

$$\widetilde{G}_+(\mathbf{p}, t, \lambda) = \big(1-\delta_{\lambda,0}\big)\exp\Big[-\rho\big(\tau+\mathbf{p}^2+L(\lambda)\big)t\Big] \tag{A.2}$$

as our Gaussian propagator. Here, $L(\lambda)$ stands for the polynomial

$$L(\lambda) = w_r\Lambda_r(\lambda) + \sum_l v_l K_l(\lambda). \tag{A.3}$$

As in section 4 it is understood that we have to set $v_l = 0$ and $\lambda = \vec{\lambda}$ or $r = 1$ and $\lambda = \overleftrightarrow{\lambda}$ to retrieve the nonlinear or the noisy RRDN, respectively.

To one-loop order the correlation function is governed by the Dyson-equation

$$\begin{aligned} G_2(\mathbf{r}, t, \lambda) - G_2^{(0)}(\mathbf{r}, t, \lambda) &= \int_{\mathbf{p}} \exp(\mathrm{i}\mathbf{p}\cdot\mathbf{r}) \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, \widetilde{G}(\mathbf{p}, t-t_1, \lambda) \\ &\quad \times \Sigma(\mathbf{p}, t_1-t_2, \lambda)\widetilde{G}(\mathbf{p}, t_2, \lambda), \end{aligned} \tag{A.4}$$

where Σ stands for the self-energy Σ depicted in figure A.1,

$$\begin{aligned} \Sigma(\mathbf{p}, s, \lambda) = \frac{\rho^2 g^2}{2}\sum_\kappa \int_{\mathbf{q}} \Big\{ &-2\exp\big(-\rho L(\lambda)s\big) \\ &+ \sum_\kappa \exp\Big[-\rho\big(L(\lambda/2+\kappa)+L(\lambda/2-\kappa)\big)s\Big]\Big\} \\ &\times \int_{\mathbf{q}} \exp\Big[-\rho\big(2\tau+(\mathbf{p}/2+\mathbf{q})^2+(\mathbf{p}/2-\mathbf{q})^2\big)s\Big]. \end{aligned} \tag{A.5}$$

Note that we have included here an extra term independent of κ to switch from the initially restricted current summation to a current summation without the restriction $\kappa \neq 0$. The integration over $\mathbf{q}$ is straightforward. After switching from the current summation to an

integration over κ we employ the saddle point method which we can since $\lambda_0 \gg 0$. From the symmetry of the integrand it follows, irrespectively of the detailed form of $L(\lambda)$, that the locus of the saddle-point is at $\kappa = 0$. Consequently, in the limit $D \to 0$,

$$\Sigma(\mathbf{p}, s, \lambda) = \frac{\rho^2 g^2}{2}\big(\exp\big(-2\rho L(\lambda/2)s\big) - 2\exp\big(-\rho L(\lambda)s\big)\big) \times \big(8\pi\rho s\big)^{-d_\perp/2}\exp\big(-\rho(2\tau + \mathbf{p}^2/2)s\big). \tag{A.6}$$

One of the time-integrations and the $\mathbf{p}$-integration in equation (A.4) are easily done. Being interested in criticality we set $\tau = 0$. For simplicity, we restrict our attention to $\mathbf{r} = \mathbf{0}$. We expand the right-hand side of equation (A.6) to first order in L and u. To prepare for the renormalization step to follow, we mark unrenormalized quantities with an open circle, $s \to \mathring{s}$, $\rho \to \mathring{\rho}$ and so on as we do in the renormalization schemes (28) and (40). Expressing various quantities through their renormalized counterparts as specified by equations (28) and (40) we arrive at

$$\mathring{G}_2(\mathbf{0}, t, \lambda)/\mathring{G}_2^{(0)}(\mathbf{0}, t, \lambda) = 1 - \frac{u(2\rho\mu^2 t)^{2-d\perp/2}}{8\Gamma(1+\varepsilon/2)} \times \int_0^1 \mathrm{d}x\, \frac{(1-x)}{\big[x(1-x/2)\big]^{d\perp/2}}\Big\{1 + x\rho t\big[2L(\lambda/2) - L(\lambda)\big]\Big\}. \tag{A.7}$$

The x-integral is performed using dimensional regularization. We obtain, up to terms of $\mathrm{O}(\varepsilon)$,

$$\mathring{G}_2(\mathbf{0}, t, \lambda)/\mathring{G}_2^{(0)}(\mathbf{0}, t, \lambda) = 1 + \frac{u(2\rho\mu^2 t)^{\varepsilon/2}}{8\Gamma(1+\varepsilon/2)} \times \Big[2 + \rho t\big(2/\varepsilon + \ln 2 - 1\big)\big[L(\lambda) - 2L(\lambda/2)\big]\Big]. \tag{A.8}$$

Equation (28) implies that the correlation function is renormalized by

$$G_2(\mathbf{0}, t, \lambda) = Z^{-1}\mathring{G}_2(\mathbf{0}, t, \lambda). \tag{A.9}$$

Using again the renormalization factors stated in equations (30) and (41) we get

$$\begin{aligned}\mathring{\rho}\mathring{L}(\lambda) &= \mathring{\rho}\mathring{w}_r\Lambda_r(\lambda) + \sum_l \mathring{\rho}\mathring{v}_l K_l(\lambda)\\ &= \rho\Big(Z^{-1}Z_{w_r}w_r\Lambda_r(\lambda) + \sum_l Z^{-1}Z_{v_l}v_l K_l(\lambda)\Big)\\ &= \rho\Big(L(\lambda) + \frac{u}{4\varepsilon}\big[L(\lambda) - 2L(\lambda/2)\big]\Big),\end{aligned} \tag{A.10}$$

where we have used the homogeneity of the polynomials $\Lambda_r(\lambda)$ and $K_l(\lambda)$. From this relation we obtain after careful expansion in u and $L(\lambda)$ to first order

$$Z^{-1}\mathring{G}_2^{(0)}(\mathbf{0}, t, \lambda) = \frac{(1-u/16)}{(4\pi\rho t)^{d_\perp/2}}\Big\{1 - \rho t\Big[L(\lambda) + \frac{u}{4\varepsilon}\big[L(\lambda) - 2L(\lambda/2)\big]\Big]\Big\}. \tag{A.11}$$

Using this result in equation (A.8) we obtain after a final ε-expansion

$$\begin{aligned}G_2(\mathbf{0}, t, \lambda) &= \frac{(1-u/16)}{(4\pi\rho t)^{d_\perp/2}}\Big\{\big(1+u/4\big)\big(1 - \rho t L(\lambda)\big) + \rho t\frac{u}{8}(\ln 2 - 1)\big[L(\lambda) - 2L(\lambda/2)\big]\\ &\quad + \rho t\frac{u}{4\varepsilon}\Big[\frac{(2\rho\mu^2 t)^{\varepsilon/2}}{\Gamma(1+\varepsilon/2)} - 1\Big]\big[L(\lambda) - 2L(\lambda/2)\big]\Big\}\\ &= \frac{(1+3u/16)}{(4\pi\rho t)^2}\Big\{1 - \rho t L(\lambda)\\ &\quad + \rho t\frac{u}{8}(\ln 2 - 1 + \ln(2\rho\mu^2 t) + C_E)[L(\lambda) - 2L(\lambda/2)]\Big\}.\end{aligned} \tag{A.12}$$

Taking into account equation (50) finally yields

$$G_2(\mathbf{0}, t_0, \lambda) = c_0\big(1 + 3u/16\big)\left\{1 - c_1\left[\big(1 + A_r u\big) w_r \Lambda_r(\lambda) + \sum_l \big(1 + A_{2l-1} u\big) v_l K_l(\lambda)\right]\right\} \tag{A.13}$$

with non-universal constants c_0 and c_1 and the amplitude A_r as stated in equation (59).

References

[1] For a review on DP see, e.g. Hinrichsen H 2001 *Adv. Phys.* **49** 815
[2] For a recent review of the field theory approach to percolation processes see Janssen H K and Täuber U C 2005 *Ann. Phys.* **315** 147
[3] For a didactic introduction and a review, see, e.g. Stauffer D and Aharony A 1994 *Introduction to Percolation Theory* (London: Taylor and Francis)
Bunde A and Havlin S (ed) 1991 *Fractals and Disordered Systems* (Berlin: Springer)
[4] Janssen H K and Stenull O 2001 *Phys. Rev.* E **63** 025103(R)
[5] Stenull O and Janssen H K 2001 *J. Stat. Phys.* **104** 1273
[6] Stenull O and Janssen H K 2001 *Phys. Rev.* E **64** 016135
[7] Stenull O and Janssen H K 2001 *Europhys. Lett.* **55** 691
[8] Stenull O and Janssen H K 2002 *Phys. Rev.* E **65** 036124
[9] Grassberger P, Hegger R and Schäfer L 1994 *J. Phys. A: Math. Gen.* **27** 7265
[10] Hager J and Schäfer L 1999 *Phys. Rev.* E **60** 2071
[11] Grassberger P, unpublished
[12] Janssen H K and Stenull O 2003 *Phys. Rev.* E **68** 036131
[13] Lübeck S and Willmann R D 2004 *J. Stat. Phys.* **115** 1231
[14] Janssen H K and Stenull O 2004 *Phys. Rev.* E **69** 016125
[15] Stenull O and Janssen H K 2003 *Phys. Rev.* E **68** 036129
[16] Redner S 1981 *J. Phys. A: Math. Gen.* **14** L349
Redner S 1982 *Phys. Rev.* B **25** 3242
[17] Redner S 1983 *Percolation Structures and Processes* ed G Deutscher *et al* (Bristol: Hilger)
[18] RRDNs were already contained implicitly in the pioneering work on percolation of Broadbent S R and Hammersley J M 1957 *Proc. Phil. Soc.* **53** 629
[19] Janssen H K 1976 *Z. Phys.* B **23** 377
Bausch R, Janssen H K and Wagner H 1976 *Z. Phys.* **24** 113
Janssen H K 1979 *Dynamical Critical Phenomena and Related Topics* (*Springer Lecture Notes in Physics* vol 104) ed C P Enz (Heidelberg: Springer)
[20] DeDominicis C 1976 *J. Physique Coll.* **37** C247
DeDominicis C and Peliti L 1978 *Phys. Rev.* B **18** 353
[21] Janssen H K 1992 *From Phase Transitions to Chaos* ed G Györgyi, I Kondor, L Sasvári and T Tél (Singapore: World Scientific)
[22] Harris A B 1987 *Phys. Rev.* B **35** 5056
[23] In the context of the random resistor network this type of generalization was proposed by Kenkel S W and Straley J P 1982 *Phys. Rev. Lett.* **49** 767
[24] Blumenfeld R and Aharony A 1985 *J. Phys. A: Math. Gen.* **18** L443
[25] This strategy was used by Park Y, Harris A B and Lubensky T C 1987 *Phys. Rev.* B **35** 5048 to treat multifractality in random resistor networks
[26] Cohn R M 1950 *Am. Math. Soc.* **1** 316
[27] See, e.g. Amit D J 1984 *Field Theory, the Renormalization Group, and Critical Phenomena* (Singapore: World Scientific)
Zinn-Justin J 1996 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)
[28] Janssen H K 1981 *Z. Phys.* B **42** 151
[29] Janssen H K 2001 *J. Stat. Phys.* **103** 801
[30] Stenull O and Janssen H K 2000 *Europhys. Lett.* **51** 539
[31] Stenull O and Janssen H K 2001 *Phys. Rev.* E **63** 036103